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Fuzzy Sumudu Transform for Solving Second Order Fuzzy Initial Value Problem under Generalized Differentiability

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Abstract:

This paper focuses on studying the solutions of second-order fuzzy initial value problems using the fuzzy Sumudu transform under the concept of generalized differentiability. The paper provides an example to illustrate the application of the proposed approach. The obtained solutions are presented in parametric form using fuzzy numbers. The results demonstrate the effectiveness of the fuzzy Sumudu transform in solving fuzzy differential equations. The conclusions highlight the advantages of the proposed method over previous approaches based on Hukuhara differentiation.

Keywords: Fuzzy numbers, Hukuhara difference, Fuzzy differential equation, Fuzzy Sumudu transform, Generalized differential.

تحويل سمودو الضبابي لحل مسألة القيمة الابتدائية الضبابية من الدرجة الثانية مع التفاضل المعمم

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الملخص:

يركز هذا البحث على دراسة حلول مسألة القيمة الابتدائية الغامضة من الدرجة الثانية باستخدام تحويل سمودو الضبابي تحت مفهوم التفاضل المعمم. وتقدم الورقة مثالًا يوضح تطبيق المنهج المقترح. يتم تقديم الحلول التي تم الحصول عليها في شكل حدودي باستخدام أرقام ضبابية. وأظهرت النتائج فعالية تحويل سمودو الضبابي في حل المعادلات التفاضلية الضبابية. ومقارنة الطريقة المقترحة بالمناهج السابقة القائمة على تمايز هوكوهارا.

الكلمات المفتاحية: الأعداد الضبابية، فرق هوكوهارا، المعادلة التفاضلية الضبابية، تحويل سمودو الضبابي، التفاضل المعمم.

Introduction:

The study of fuzzy differential equations (FDEs) provides a suitable framework for mathematical modeling of real-world problems where uncertainty or ambiguity prevails. Most practical problems can be modeled as FDEs. Therefore, FDEs play a crucial role in both theory and practical implementation. They have wide-ranging applications in diverse fields, including population modeling, engineering, chaotic systems, and hydraulic modeling. The differentiability of functions with fuzzy values was first introduced by Change and Zadeh [1] and then Dubois and Prade [2] defined and used the expansion principle. Other methods are discussed by Puri and Ralescu[3] who generalized and extended the concept of Hukuhara differentiability for set-valued mappings to a class of fuzzy mappings.

In the last few years, many researchers have worked on the theory of fuzzy differential equations [4], [5] and other recent works, such as the study of some topological properties and structure of the solutions to the Cauchy problem for fuzzy differential systems (see [6], [7]]). Subsequently, some significant extensions of the fuzzy differential

equations based on H-derivative are the fuzzy functional differential equations [8], the random fuzzy differential equations [9], and the fuzzy neutral differential equations [10]. However, the approach that employs Hukuhara differentiation encounters a significant drawback. Specifically, the solution progressively becomes more indistinct as time elapses, making it exceedingly challenging to derive profound outcomes regarding the qualitative theory for fuzzy differential equations. These outcomes may include properties like asymptotic behavior, periodicity, and bifurcation.

The strongly generalized differentiability was introduced in [11] and studied in [12], [13]. This concept allows us to resolve the shortcoming mentioned above. Indeed, the strongly generalized derivative is defined for a larger class of fuzzy number valued functions than the Hukuhara derivative. Hence, we use this differentiability concept in the present paper.

The fuzzy Sumudu transform is an effective method for solving fuzzy differential equations. Applying the fuzzy Sumudu transform directly makes it possible to find the solution of the fuzzy differential equation that meets the given initial condition. The concept of the fuzzy Sumudu transform was initially introduced by Abdul Rahman and Ahmed [14]. Subsequently, many research papers have utilized the fuzzy Sumudu transform to investigate the solution of fuzzy differential equations [15], [16].

In this work, we investigate the solutions to the problem

$$\delta y''(x) + \beta y'(x) = [0]_\alpha \ , x > 0 \qquad (1)$$

$$y(0) = [A]_\alpha \ , \ y'(0) = [B]_\alpha \qquad (2)$$

by the fuzzy Sumudu transform under the concept of generalized differentiability, where $\delta, \beta > 0, [0]_\alpha = [-1+\alpha, 1-\alpha]$, a and b are symmetric triangular fuzzy numbers with supports $[\underline{a}_\alpha, \overline{a}_\alpha], [\underline{b}_\alpha, \overline{b}_\alpha]$ and the α-level sets of a, b are

$$[A]_\alpha = [\underline{A}_\alpha, \overline{A}_\alpha] = \left[\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right],$$

$$[B]_\alpha = [\underline{B}_\alpha, \overline{B}_\alpha] = \left[\underline{b} + \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha\right].$$

The structure of this paper is as follows: Section provides an overview of the preliminary concepts, presents the main results, and Section concludes the paper.

Preliminaries:

The following concepts and definitions are useful in the given work:

Definition 2.1. [17] A fuzzy number is a mapping $u: \mathbb{R} \rightarrow [0,1]$ with the following criteria.

1. u is normal, i.e., there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

2. u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$, for all $\lambda \in [0,1], x, y \in \mathbb{R}$.

3. u is upper semi continuous, i.e., for any $x_0 \in \mathbb{R}$,

$$u(x_0) \geq \lim_{x \to x_0^{\pm}} u(x).$$

4. Supp $u = \{x \in \mathbb{R}: u(x) > 0\}$ is the support of u, and its closure cl (supp u) is compact.

Definition 2.2. [17] Let u be a fuzzy number defined in $\mathcal{F}(\mathbb{R})$. The α-level set of u, for any $\alpha \in [0,1]$,denoted by $[u]_\alpha$, is a crisp set that contains all elements in $\mathbb{R}$, such that the membership value of u is greater or equal to ∝, that is

$$[u]_\alpha = \{x \in \mathbb{R}: u(x) \geq \alpha\} \quad (3)$$

Whenever we represent the fuzzy number with α-level set, we can see that it is closed and bounded. It is denoted by $[\underline{u}_\alpha, \overline{u}_\alpha]$, where they represent the lower and upper bound α-level set of a fuzzy number, respectively.

As the fuzzy number is resolved by the interval $[u]_\alpha$, researchers[18],[19] defined another representation, parametrically, of fuzzy numbers as in the following definition.

Definition 2.3. [20] A fuzzy number u in parametric form is a pair $[\underline{u}_\alpha, \overline{u}_\alpha]$ of functions $\underline{u}_\alpha$ and $\overline{u}_\alpha$ for any $\alpha \in [0,1]$, which satisfies the following requirements.

1. $\underline{u}_\alpha$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuous at 0.

2. $\overline{u}_\alpha$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuous at 0.

3. $\underline{u}_\alpha \leq \overline{u}_\alpha$

Definition 2.4. [21] The α-level set of symmetric triangular fuzzy number A with support $[\underline{a}_\alpha, \overline{a}_\alpha]$ is:

$$[A]_\alpha = [\underline{A}_\alpha, \overline{A}_\alpha] = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right] \quad (4)$$

Definition 2.5. [19] Let $[u]_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha], [v]_\alpha = [\underline{v}_\alpha, \overline{v}_\alpha]$ and $k \in \mathbb{R}$, the operations addition, subtraction, multiplication and scalar multiplication are defined as

$$[u \oplus v]_\alpha = [u]_\alpha + [v]_\alpha = [\underline{u}_\alpha + \underline{v}_\alpha, \overline{u}_\alpha + \overline{v}_\alpha] \quad (5)$$

$$[u \ominus v]_\alpha = [u]_\alpha - [v]_\alpha = [\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha] \quad (6)$$

$$[u \odot v]_\alpha = \begin{pmatrix} \min\{\underline{u}_\alpha \underline{v}_\alpha, \underline{u}_\alpha \overline{v}_\alpha, \overline{u}_\alpha \underline{v}_\alpha, \overline{u}_\alpha \overline{v}_\alpha\} \\ max\{\underline{u}_\alpha \underline{v}_\alpha, \underline{u}_\alpha \overline{v}_\alpha, \overline{u}_\alpha \underline{v}_\alpha, \overline{u}_\alpha \overline{v}_\alpha\} \end{pmatrix} \quad (7)$$

$$[ku]_\alpha = k[u]_\alpha = \begin{cases} (k\underline{u}_\alpha, k\overline{u}_\alpha), & k \geq 0 \\ (k\overline{u}_\alpha, k\underline{u}_\alpha), & k \leq 0 \end{cases} \quad (8)$$

Definition 2.6. [22] For arbitrary fuzzy numbes $[u]_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha], [v]_\alpha = [\underline{v}_\alpha, \overline{v}_\alpha]$ the quantity

$$d(u, v) = \sup_{0 \leq \alpha \leq 1} \max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{v}_\alpha|\}$$

Is the Hausdroff distance between u and v.

Definition 2.7. [23] Let $f: \mathbb{R} \to \mathcal{F}(\mathbb{R})$ and it is represented by $[\underline{f_\alpha}(x), \overline{f}_\alpha(x)]$. For any fixed $\alpha \in [0,1]$, assume $\underline{f_\alpha}(x)$ and $\overline{f}_\alpha(x)$ are Riemann-integrable on $[a, b]$ for every $b \geq a$, and assume there are ywo positive $\underline{M_\alpha}$ and $\overline{M}_\alpha$ such that $\int_a^b |\underline{f_\alpha}(x)| dx \leq \underline{M_\alpha}$ and $\int_a^b |\overline{f}_\alpha(x)| dx \leq \overline{M}_\alpha$ for every $b \geq a$. Then, $f(x)$ is improper fuzzy Riemann-integrable on $[a, \infty)$ and the improper fuzzy Riemann-integrable is a fuzzy number. Further more, we have

$$\int_a^\infty f(x)dx = [\int_a^\infty \underline{f_\alpha}_\alpha(x)dx, \int_a^\infty \overline{f}_\alpha(x)dx] \quad (9)$$

Definition 2.8. [3] If $u, v \in \mathcal{F}(\mathbb{R})$ and if there exists a fuzzy subset $k \in \mathcal{F}(\mathbb{R})$ such that $k + u = v$ is unique. In this case, k is called the Hukuhara difference, or simply H-difference of u and v denoted by $v -^H u$.

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Definition 2.9. [11] Let $f: \mathbb{R} \to \mathcal{F}(\mathbb{R})$, and $x_0 \in (a, b)$. Then f is said to be strongly generalized differentiable at x_0, if there exists an element $f'(x) \in \mathcal{F}(\mathbb{R})$, such that

1. For all $h > 0$ sufficiently small, there exist $f(x_0 + h) -^H f(x_0), f(x_0) -^H f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f(x_0+h) -^H f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0) -^H f(x_0-h)}{h} = f'(x) \quad (10)$$

2. For all $h > 0$ sufficiently small, there exist $f(x_0) -^H f(x_0 + h), f(x_0 - h) -^H f(x_0)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f(x_0) -^H f(x_0+h)}{-h} = \lim_{h\to 0} \frac{f(x_0-h) -^H f(x_0)}{-h} = f'(x) \quad (11)$$

3. For all $h > 0$ sufficiently small, there exist $f(x_0 + h) -^H f(x_0), f(x_0 - h) -^H f(x_0)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f(x_0+h) -^H f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0-h) -^H f(x_0)}{-h} = f'(x) \quad (12)$$

4. For all $h > 0$ sufficiently small, there exist $f(x_0) -^H f(x_0 + h), f(x_0) -^H f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f(x_0) -^H f(x_0+h)}{-h} = \lim_{h\to 0} \frac{f(x_0-h) -^H f(x_0-h)}{h} = f'(x) \quad (13)$$

In this paper, we only consider case (1) and (2) in the strongly generalized differentiability proposed by Beda and Gal[11] since they are more important as stated in [24] ,[12].

Theorem 2.1. [25] Let $f: \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued and denote $f(x) = \left[\underline{f_\alpha}(x), \overline{f}_\alpha(x)\right]$, foreach $\alpha \in [0,1]$. Then

1. If is (1)-differentiable, then $\underline{f_\alpha}(x)$ and $\overline{f}_\alpha(x)$ are differentiable functions and $f'(x) = [\underline{f_\alpha'}(x), \overline{f}'_\alpha(x)]$.
2. If is (2)-differentiable, then $\underline{f_\alpha}(x)$ and $\overline{f}_\alpha(x)$ are differentiable functions and $f'(x) = [\overline{f}'_\alpha(x), \underline{f_\alpha'}(x)]$.

Definition 2.10. [11] We say that a mapping $f: \mathbb{R} \to \mathcal{F}(\mathbb{R})$ is strongly generalized differentiable of the second-order at $x_0 \in (a, b)$, if there exists an element $f''(x) \in \mathcal{F}(\mathbb{R})$, such that

2. For all $h > 0$ sufficiently small, there exist $f'(x_0 + h) -^H f'(x_0), f'(x_0) -^H f'(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f'(x_0+h) -^H f'(x_0)}{h} = \lim_{h\to 0} \frac{f'(x_0) -^H f'(x_0-h)}{h} = f''(x) \quad (14)$$

For all $h > 0$ sufficiently small, there exist $f'(x_0) -^H f'(x_0 + h), f'(x_0 - h) -^H f'(x_0)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f'(x_0) -^H f'(x_0+h)}{-h} = \lim_{h\to 0} \frac{f'(x_0-h) -^H f'(x_0)}{-h} = f''(x) \quad (15)$$

For all $h > 0$ sufficiently small, there exist $f'(x_0 + h) -^H f'(x_0), f'(x_0 - h) -^H f'(x_0)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f'(x_0+h) -^H f'(x_0)}{h} = \lim_{h\to 0} \frac{f'(x_0-h) -^H f'(x_0)}{-h} = f''(x) \quad (16)$$

For all $h > 0$ sufficiently small, there exist $f'(x_0) -^H f'(x_0 + h), f'(x_0) -^H f'(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{f'(x_0) -^H f'(x_0+h)}{-h} = \lim_{h\to 0} \frac{f'(x_0-h) -^H f'(x_0-h)}{h} = f''(x) \quad (17)$$

Theorem 2.2. [25] Let $f': \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued and denote $f(x) = \left[\underline{f}_\alpha(x), \overline{f}_\alpha(x)\right]$, foreach $\alpha \in [0,1]$, the function f is (1)-differentiable or (2)-differentiable.

3. If f and f' are (1)-differentiable, then $\underline{f'}_\alpha(x)$ and $\overline{f'}_\alpha(x)$ are differentiable functions and $f''(x) = [\underline{f}''_\alpha(x), \overline{f}''_\alpha(x)]$.

If f is (1)-differentiable, and f' is (2)-differentiable, then $\underline{f}_\alpha(x)$ and $\overline{f}_\alpha(x)$ are differentiable functions and $f''(x) = [\overline{f}''_\alpha(x), \underline{f}''_\alpha(x)]$.

If f is (2)-differentiable, and f' is (1)-differentiable, then $\underline{f}_\alpha(x)$ and $\overline{f}_\alpha(x)$ are differentiable functions and $f''(x) = [\overline{f}''_\alpha(x), \underline{f}''_\alpha(x)]$.

If f and f' are (2)-differentiable, then $\underline{f'}_\alpha(x)$ and $\overline{f'}_\alpha(x)$ are differentiable functions and $f''(x) = [\underline{f}''_\alpha(x), \overline{f}''_\alpha(x)]$.

Definition 2.11. [12] [10] Let $f: \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be continuous fuzzy-valued function defined in parametric form $f(x) = \left[\underline{f}_\alpha(x), \overline{f}_\alpha(x)\right]$ for $0 \leq \alpha \leq 1$. Suppose that $f(ux) \odot e^{-x}$ is improper fuzzy Riemann integrable function on $[0, \infty)$, with $u > 0$ a real parameter, then

$$S[f(x)] = \int_0^\infty f(ux) \odot e^{-x} dx = G(u) \tag{18}$$

Is called fuzzy Sumudu transform. Since f is fuzzy-valued function, therefore parametric representation of Eq. (10) willbe, for $0 \leq \alpha \leq 1$:

$$\int_0^\infty f(ux) \odot e^{-x} dx = \left[\int_a^\infty \underline{f}_\alpha(ux) e^{-x} dx, \int_a^\infty \overline{f}_\alpha(ux) e^{-x} dx\right] \tag{19}$$

or

$$S[f(x)] = \left[S\left[\underline{f}_\alpha(x)\right], S[\overline{f}_\alpha(x)]\right] = [\underline{G}(u), \overline{G}(u)] \tag{20}$$

were

$$\underline{G}(u) = S\left[\underline{f}_\alpha(x)\right] = \int_a^\infty \underline{f}_\alpha(ux) e^{-x} dx$$

and

$$\overline{G}(u) = S[\overline{f}_\alpha(x)] = \int_a^\infty \overline{f}_\alpha(ux) e^{-x} dx \tag{21}$$

Theorem 2.3. [14] Assume that the $f(x)$ is a continuous fuzzy-valued function on $[0, \infty)$, also $k \geq 0$, thus

$$S[k \odot f(x)] = k \odot S[f(x)] \tag{22}$$

Theorem 2.4. [14] Let $f, g: \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be two continuous fuzzy-valued functions. Suppose that c_1 and c_2 are arbitrary constants, then

$$S[(c_1 \odot f(x)) \oplus (c_2 \odot g(x))] = (c_1 \odot S[f(x)]) \oplus (c_2 \odot S[g(x)]).$$

Theorem 2.5. [14] Suppose $f(x)$ is a continuous fuzzy-valued function on $[0, \infty)$, then

$$S\left[x \odot \frac{df(x)}{dx}\right] = u\frac{dG(u)}{du} \tag{23}$$

Theorem 2.6. [16] Suppose $f'(x)$ be a fuzzy value integrable function, as well as $f(x)$ be the primitive of $f'(x)$ on $[0, \infty)$. Therefore,

$$S[f'(x)] = \left(\frac{1}{u} \odot S[f(x)]\right) \ominus \left(\frac{1}{u} \odot f(0)\right) \tag{24}$$

where f is considered to be (1)-differentiable, or

$$S[f'(x)] = \left(\frac{-1}{u} \odot f(0)\right) \ominus \left(\frac{-1}{u} \odot S[f(x)]\right) \tag{25}$$

where f is considered to be (2)-differentiable.

Theorem 2.7. [16] Suppose $f''(x)$ be a fuzzy value integrable function, as well as $f(x)$ be the primitive of $f''(x)$ on $[0, \infty)$. Therefore,

$$S[f''(x)] = \left(\frac{1}{u^2} \odot S[f(x)]\right) \ominus \left(\frac{1}{u^2} \odot f(0)\right) \ominus \left(\frac{1}{u} \odot f'(0)\right) \tag{26}$$

where f and f' are (1)-differentiable, or

$$S[f''(x)] = \left(\frac{-1}{u} \odot f'(0)\right) \ominus \left(\frac{-1}{u^2} \odot S[f(x)]\right) - \left(\frac{1}{u^2} \odot f(0)\right) \tag{27}$$

where f is (1)-differentiable and f' is (2)-differentiable, or

$$S[f''(x)] = \left(\frac{-1}{u^2} \odot f(0)\right) \ominus \left(\frac{-1}{u^2} \odot S[f(x)]\right)$$

$$\ominus \left(\frac{1}{u} \odot f'(0)\right) \tag{28}$$

where f is (2)-differentiable and f' is (1)-differentiable, or

$$S[f''(x)] = \left(\frac{1}{u^2} \odot S[f(x)]\right) \ominus \left(\frac{1}{u^2} \odot f(0)\right)$$

$$-\left(\frac{1}{u} \odot f'(0)\right) \tag{29}$$

where f and f' are (2)-differentiable.

Main Results:

In this section, we research the solutions to problems by fuzzy Sumudu transform under the concept of generalized differentiability In this paper, (i,j) solution means that y is (i)-differentiable, y' is (j)-differentiable, i,j=1,2.

3.1. (1,1) solution of the problem:

If y and y' are (1)-differentiable, since

$$S[[0]_\alpha] = \delta\left(\left(\frac{1}{u^2} \odot S[y(x)]\right) \ominus \left(\frac{1}{u^2} \odot y(0)\right) \ominus \left(\frac{1}{u} \odot y'(0)\right)\right) + \beta\left(\left(\frac{1}{u} \odot S[y(x)]\right) \ominus \left(\frac{1}{u} \odot y(0)\right)\right), \tag{30}$$

we have the equations

$$S[-1+\alpha] = \frac{\delta}{u^2} S\left[\underline{y}_\alpha(x)\right] - \frac{\delta}{u^2} \underline{y}_\alpha(0) - \frac{\delta}{u} \underline{y}'_\alpha(0) + \frac{\beta}{u} S\left[\underline{y}_\alpha(x)\right] - \frac{\beta}{u} \underline{y}_\alpha(0), \tag{31}$$

$$S[1-\alpha] = \frac{\delta}{u^2} S[\overline{y}_\alpha(x)] - \frac{\delta}{u^2} \overline{y}_\alpha(0) - \frac{\delta}{u} \overline{y}'_\alpha(0) + \frac{\beta}{u} S[\overline{y}_\alpha(x)]$$

$$-\frac{\beta}{u}\overline{y}_{\alpha}(0). \quad (32)$$

Using the initial values (2), we get

$$S\left[\underline{y}_{\alpha}(x)\right]\left(\frac{\delta+\beta u}{u^2}\right) = (-1+\alpha) + \frac{\delta}{u}\underline{B}_{\alpha} + \left(\frac{\delta}{u^2}+\frac{\beta}{u}\right)\underline{A}_{\alpha}, \quad (33)$$

$$S[\overline{y}_{\alpha}(x)]\left(\frac{\delta+\beta u}{u^2}\right) = (1-\alpha) + \frac{\delta}{u}\overline{B}_{\alpha} + \left(\frac{\delta}{u^2}+\frac{\beta}{u}\right)\overline{A}_{\alpha}. \quad (34)$$

From this, we obtain

$$S\left[\underline{y}_{\alpha}(x)\right] = \frac{u^2}{\delta+\beta u}(-1+\alpha) + \frac{\delta u}{\delta+\beta u}\underline{B}_{\alpha} + \underline{A}_{\alpha}, \quad (35)$$

$$S[\overline{y}_{\alpha}(x)] = \frac{u^2}{\delta+\beta u}(1-\alpha) + \frac{\delta u}{\delta+\beta u}\overline{B}_{\alpha} + \overline{A}_{\alpha}. \quad (36)$$

Now, taking the inverse Sumudu transform of the above equations, (1,1) solution is obtained as

$$\underline{y}_{\alpha}(x) = (-1+\alpha)\left(\frac{x}{\beta} + \frac{\delta}{\beta^2}\left(e^{-\frac{\beta}{\delta}x} - 1\right)\right) - \frac{\delta \underline{B}_{\alpha}}{\beta}\left(e^{-\frac{\beta}{\delta}x} - 1\right) + \underline{A}_{\alpha}, \quad (37)$$

$$\overline{y}_{\alpha}(x) = (1-\alpha)\left(\frac{x}{\beta} + \frac{\delta}{\beta^2}\left(e^{-\frac{\beta}{\delta}x} - 1\right)\right) - \frac{\delta \overline{B}_{\alpha}}{\beta}\left(e^{-\frac{\beta}{\delta}x} - 1\right) + \overline{A}_{\alpha}, \quad (38)$$

$$[y(x)]_{\alpha} = \left[\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right].$$

3.2. (1,2) solution of the problem:

If y is (1)-differentiable and y' is (2)-differentiable, We have the equations

$$S[[0]_{\alpha}] = \delta\left(\left(\frac{-1}{u} \odot f'(0)\right) \ominus \left(\frac{-1}{u^2} \odot S[f(x)]\right) - \left(\frac{1}{u^2} \odot f(0)\right)\right) + \beta\left(\left(\frac{1}{u} \odot S[y(x)]\right) \ominus \left(\frac{1}{u} \odot y(0)\right)\right), \quad (39)$$

$$S[-1+\alpha] = -\frac{\delta}{u}\overline{y}'_{\alpha}(0) + \frac{\delta}{u^2}S[\overline{y}_{\alpha}(x)] - \frac{\delta}{u^2}\overline{y}_{\alpha}(x) + \frac{\beta}{u}S\left[\underline{y}_{\alpha}(x)\right] - \frac{\beta}{u}\underline{y}_{\alpha}(0) \quad (40)$$

$$S[1-\alpha] = -\frac{\delta}{u}\underline{y}'_{\alpha}(0) + \frac{\delta}{u^2}S\left[\underline{y}_{\alpha}(x)\right] - \frac{\delta}{u^2}\underline{y}_{\alpha}(0) + \frac{\beta}{u}S[\overline{y}_{\alpha}(x)] - \frac{\beta}{u}\overline{y}_{\alpha}(0). \quad (41)$$

From this, we obtain the equation

$$\frac{\delta}{u^2}S[\overline{y}_{\alpha}(x)] + \frac{\beta}{u}S\left[\underline{y}_{\alpha}(x)\right] = (-1+\alpha) + \frac{\delta}{u}\overline{B}_{\alpha} + \frac{\delta}{u^2}\overline{A}_{\alpha} + \frac{\beta}{u}\underline{A}_{\alpha}, \quad (42)$$

$$\frac{\delta}{u^2}S\left[\underline{y}_{\alpha}(x)\right] + \frac{\beta}{u}S[\overline{y}_{\alpha}(x)] = (1-\alpha) + \frac{\delta}{u}\underline{B}_{\alpha} + \frac{\delta}{u^2}\underline{A}_{\alpha} + \frac{\beta}{u}\overline{A}_{\alpha}, \quad (43)$$

If $S[\overline{y}_{\alpha}(x)]$ in the equation (42) is replaced by the equation (43), we have

$$S\left[\underline{y}_{\alpha}(x)\right] = \frac{u^2}{\delta-\beta u}(1-\alpha) + \frac{\delta^2 u}{\delta^2-\beta^2u^2}\underline{B}_{\alpha} - \frac{\alpha\beta u^2}{\delta^2-\beta^2u^2}\overline{B}_{\alpha} + \underline{A}_{\alpha}, \quad (44)$$

Now, taking the inverse Sumudu transform of the above equations, (44) solution is obtained as

$$\underline{y}_{\alpha}(x) = (1-\alpha)\left(-\frac{x}{\beta} + \frac{\delta}{\beta^2}\left(e^{\frac{\beta}{\delta}x} - 1\right)\right) - \overline{B}_{\alpha}\left(\frac{\delta}{2\beta}\left(e^{\frac{\beta}{\delta}x} + e^{-\frac{\beta}{\delta}x} - 2\right)\right) + \underline{B}_{\alpha}\left(\frac{\delta\underline{B}_{\alpha}}{2\beta}\left(e^{\frac{\beta}{\delta}x} - e^{-\frac{\beta}{\delta}x}\right)\right) + \underline{A}_{\alpha}, \quad (45)$$

Similarly, the upper solution is obtained as

$$\overline{y}_{\alpha}(x) = (-1+\alpha)\left(-\frac{x}{\beta} + \frac{\delta}{\beta^2}\left(e^{\frac{\beta}{\delta}x} - 1\right)\right) - \underline{B}_{\alpha}\left(\frac{\delta\underline{B}_{\alpha}}{2\beta}\left(e^{\frac{\beta}{\delta}x} + e^{-\frac{\beta}{\delta}x} - 2\right)\right) + \overline{B}_{\alpha}\left(\frac{\delta}{2\beta}\left(e^{\frac{\beta}{\delta}x} - e^{-\frac{\beta}{\delta}x}\right)\right) + \overline{A}_{\alpha}. \quad (46)$$

3.3. (2,1) solution of the problem:

If y is (2)-differentiable and y' is (1)-differentiable, We have the equations

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$$S[[0]_\alpha]=\delta\left(\left(\frac{-1}{u^2}\odot f(0)\right)\ominus\left(\frac{-1}{u^2}\odot S[f(x)]\right)\ominus\left(\frac{1}{u}\odot\ f'(0)\right)\right)+\beta\left(\left(\frac{-1}{u}\odot f(0)\right)\ominus\left(\frac{-1}{u}\odot S[f(x)]\right)\right), \quad (47)$$

We have the equations

$$S[-1+\alpha]=-\frac{\delta}{u^2}\overline{y}_\alpha(0)+\frac{\delta}{u^2}S[\overline{y}_\alpha(x)]-\frac{\delta}{u}\overline{y}'_\alpha(0)-\frac{\beta}{u}\overline{y}_\alpha(0)+\frac{\beta}{u}S[\overline{y}_\alpha(x)], \quad (48)$$

$$S[1-\alpha]=-\frac{\delta}{u^2}\underline{y}_\alpha(0)+\frac{\delta}{u^2}S\left[\underline{y}_\alpha(x)\right]-\frac{\delta}{u}\underline{y}'_\alpha(0)-\frac{\beta}{u}\underline{y}_\alpha(0)+\frac{\beta}{u}S\left[\underline{y}_\alpha(x)\right]. \quad (49)$$

That is,

$$S\left[\underline{y}_\alpha(x)\right]=\frac{u^2}{\delta+\beta u}(-1+\alpha)+\frac{\delta u}{\delta+\beta u}\overline{B}_\alpha+\underline{A}_\alpha, \quad (50)$$

$$S[\overline{y}_\alpha(x)]=\frac{u^2}{\delta+\beta u}(1-\alpha)+\frac{\delta u}{\delta+\beta u}\underline{B}_\alpha+\overline{A}_\alpha. \quad (51)$$

From this, we obtain

$$\underline{y}_\alpha(x)=(1-\alpha)\left(\frac{x}{\beta}-\frac{\delta}{\beta^2}\left(1-e^{-\frac{\beta}{\delta}x}\right)\right)+\frac{\delta\overline{B}_\alpha}{\beta}\left(1-e^{-\frac{\beta}{\delta}x}\right)+\underline{A}_\alpha, \quad (52)$$

$$\overline{y}_\alpha(x)=(-1+\alpha)\left(\frac{x}{\beta}-\frac{\delta}{\beta^2}\left(1-e^{-\frac{\beta}{\delta}x}\right)\right)+\frac{\delta\underline{B}_\alpha}{\beta}\left(1-e^{-\frac{\beta}{\delta}x}\right)+\overline{A}_\alpha, \quad (53)$$

$$[y(x)]_\alpha=\left[\underline{y}_\alpha(x),\overline{y}_\alpha(x)\right].$$

3.4. (2,2) solution of the problem:

If y and y' are (2)-differentiable, since

$$S[[0]_\alpha]=\delta\left(\left(\frac{1}{u^2}\odot S[f(x)]\right)\ominus\left(\frac{1}{u^2}\odot f(0)\right)-\left(\frac{1}{u}\odot f'(0)\right)\right)+\beta\left(\left(\frac{-1}{u}\odot f(0)\right)\ominus\quad\left(\frac{-1}{u}\odot S[f(x)]\right)\right), \quad (54)$$

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the equations

$$S[-1+\alpha] = \frac{\delta}{u^2} S\left[\underline{y}_\alpha(x)\right] - \frac{\delta}{u^2}\underline{y}_\alpha(0) - \frac{\delta}{u}\underline{y}'_\alpha(0) - \frac{\beta}{u}\overline{y}_\alpha(0) + \frac{\beta}{u} S[\overline{y}_\alpha(x)], \quad (55)$$

$$S[1-\alpha] = \frac{\delta}{u^2} S[\overline{y}_\alpha(x)] - \frac{\delta}{u^2}\overline{y}_\alpha(0) - \frac{\delta}{u}\overline{y}'_\alpha(0) - \frac{\beta}{u}\underline{y}_\alpha(0) + \frac{\beta}{u} S\left[\underline{y}_\alpha(x)\right]. \quad (56)$$

From this, we obtain the equation

$$\frac{\delta}{u^2} S\left[\underline{y}_\alpha(x)\right] + \frac{\beta}{u} S[\overline{y}_\alpha(x)] = (-1+\alpha) + \frac{\delta}{u^2}\underline{A}_\alpha + \frac{\delta}{u}\overline{B}_\alpha + \frac{\beta}{u}\overline{A}_\alpha, \quad (57)$$

$$\frac{\delta}{u^2} S[\overline{y}_\alpha(x)] + \frac{\beta}{u} S\left[\underline{y}_\alpha(x)\right] = (1-\alpha) + \frac{\delta}{u^2}\overline{A}_\alpha + \frac{\delta}{u}\underline{B}_\alpha + \frac{\beta}{u}\underline{A}_\alpha. \quad (58)$$

If $S[\overline{y}_\alpha(x)]$ in the equation (58) is replaced by the equation (57), we have

$$S\left[\underline{y}_\alpha(x)\right] = \frac{u^2}{\delta-\beta u}(1-\alpha) + \frac{\delta^2 u}{\delta^2-\beta^2u^2}\overline{B}_\alpha - \frac{\alpha\beta u^2}{\delta^2-\beta^2u^2}\underline{B}_\alpha + \underline{A}_\alpha, \quad (59)$$

Taking the inverse Sumudu transform of the above equations (57), the lower solution is obtained as

$$\underline{y}_\alpha(x) = (-1+\alpha)\left(\frac{\delta}{\beta^2}\left(e^{\frac{\beta}{\delta}x} - 1\right) - \frac{x}{\beta}\right)$$

$$-\underline{B}_\alpha\left(\frac{\delta}{2\beta}\left(e^{\frac{\beta}{\delta}x} + e^{-\frac{\beta}{\delta}x} - 2\right)\right) + \overline{B}_\alpha\left(\frac{\delta}{2\beta}\left(e^{\frac{\beta}{\delta}x} - e^{-\frac{\beta}{\delta}x}\right)\right) + \underline{A}_\alpha, \quad (60)$$

Similarly, the upper solution is obtained as

$$\overline{y}_\alpha(x) = (1-\alpha)\left(\frac{\delta}{\beta^2}\left(e^{\frac{\beta}{\delta}x} - 1\right) - \frac{x}{\beta}\right)$$

$$-\overline{B}_\alpha\left(\frac{\delta}{2\beta}\left(e^{\frac{\beta}{\delta}x}+e^{-\frac{\beta}{\delta}x}-2\right)\right)+\underline{B}_\alpha\left(\frac{\delta}{2\beta}\left(e^{\frac{\beta}{\delta}x}-e^{-\frac{\beta}{\delta}x}\right)\right)+\overline{A}_\alpha. \qquad (61)$$

Examples 1. Consider the solutions of the problem

$$2y''(x)+y'(x)=[0]_\alpha, \qquad x>0$$

$$y(0)=(-1+\alpha,1-\alpha)$$

$$y'(0)=(\alpha,2-\alpha)$$

By fuzzy Sumudu transform

Solve:

(1,1) solution is

$$\underline{y}_\alpha(x)=(-1+\alpha)\left(x+2e^{-\frac{x}{2}}\right)+\alpha\left(1-2e^{-\frac{x}{2}}\right)+1$$

$$=\alpha(x+1)+1-x-2e^{-\frac{x}{2}}$$

$$\overline{y}_\alpha(x)=(1-\alpha)\left(x+2e^{-\frac{x}{2}}\right)-2\left(2e^{-\frac{x}{2}}-\alpha e^{-\frac{x}{2}}\right)-\alpha+3$$

$$=-\alpha(1+x)+x-2e^{-\frac{x}{2}}+3$$

(1,2) solution is

$$\underline{y}_\alpha(x)=(1-\alpha)\left(2e^{\frac{x}{2}}-x\right)-(2-\alpha)\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)+\alpha\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)$$
$$+1+\alpha$$

$$=\alpha(x+1)+1-x-2e^{-\frac{x}{2}}$$

$$\overline{y}_\alpha(x)=(-1+\alpha)\left(2e^{\frac{x}{2}}-x\right)-\alpha\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)+(2-\alpha)\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)$$
$$+3-\alpha$$

$$=-\alpha(1+x)+x-2e^{-\frac{x}{2}}+3$$

If

$$\frac{\partial \underline{y}_\alpha(x)}{\partial \alpha} \geq 0, \frac{\partial \overline{y}_\alpha(x)}{\partial \alpha} \leq 0, \underline{y}_\alpha(x) \leq \overline{y}_\alpha(x)$$

$$\underline{y}'_\alpha(x) \leq \overline{y}'_\alpha(x), \underline{y}''_\alpha(x) \leq \overline{y}''_\alpha(x),$$

(1,1) solution is a valid fuzzy function. If

$$\frac{\partial \underline{y}_\alpha(x)}{\partial \alpha} \geq 0, \frac{\partial \overline{y}_\alpha(x)}{\partial \alpha} \leq 0, \underline{y}_\alpha(x) \leq \overline{y}_\alpha(x)$$

$$\underline{y}'_\alpha(x) \leq \overline{y}'_\alpha(x), \underline{y}''_\alpha(x) \leq \overline{y}''_\alpha(x),$$

(1,2) solution is a valid fuzzy function. For (1,1) solution, since

$$\frac{\partial \underline{y}_\alpha(x)}{\partial \alpha} = x + 1 > 0, \frac{\partial \overline{y}_\alpha(x)}{\partial \alpha} = -(1 + x) < 0,$$

$$\overline{y}_\alpha(x) - \underline{y}_\alpha(x) = (1 - \alpha)(2 + x) \geq 0$$

$$\overline{y}'_\alpha(x) - \underline{y}'_\alpha(x) = 2(1 - \alpha) \geq 0, \overline{y}''_\alpha(x) - \underline{y}''_\alpha(x) = 0$$

(1,1) , (1,2) solutions are a valid fuzzy functions. Also, for (1,1) and (1,2) solutions, since

$$\underline{y}_1(x) = 2 - 2e^{-\frac{x}{2}} = \overline{y}_1(x),$$

$$\underline{y}_1(x) - \underline{y}_\alpha(x) = (1 + x)(1 - \alpha) = \overline{y}_\alpha(x) - \overline{y}_1(x),$$

(1,1) and (1,2) solutions are symmetric triangular fuzzy numbers for any $x > 0$ time.

(2,1) solution is

$$\underline{y}_\alpha(x) = \alpha(1 - x) + 1 + x - 2e^{-\frac{x}{2}},$$

$$\overline{y}_\alpha(x) = \alpha(x - 1) - x - 2e^{-\frac{x}{2}} + 3$$

(2,2) solution is

$\underline{y}_\alpha(x) = \alpha(1-x)+1+x-2e^{-\frac{x}{2}},$

$\overline{y}_\alpha(x) = \alpha(x-1)-x-2e^{-\frac{x}{2}}+3$

$$[y(x)]_\alpha = \left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right]$$

If

$$\frac{\partial \underline{y}_\alpha(x)}{\partial \alpha} \geq 0, \frac{\partial \overline{y}_\alpha(x)}{\partial \alpha} \leq 0, \underline{y}_\alpha(x) \leq \overline{y}_\alpha(x)$$

$$\overline{y}_\alpha{}'(x) \leq \underline{y}_\alpha{}'(x), \overline{y}_\alpha{}''(x) \leq \underline{y}_\alpha{}''(x),$$

(2,1) solution is a valid fuzzy function. If

$$\frac{\partial \underline{y}_\alpha(x)}{\partial \alpha} \geq 0, \frac{\partial \overline{y}_\alpha(x)}{\partial \alpha} \leq 0, \underline{y}_\alpha(x) \leq \overline{y}_\alpha(x),$$

$$\overline{y}'_\alpha(x) \leq \underline{y}'_\alpha(x), \overline{y}''_\alpha(x) \leq \underline{y}''_\alpha(x),$$

(2,2) solution is a valid fuzzy function. For (2,1) solution, since

$$\frac{\partial \underline{y}_\alpha(x)}{\partial \alpha} = 1-x, \frac{\partial \overline{y}_\alpha(x)}{\partial \alpha} = x-1,$$

$$\overline{y}_\alpha(x) - \underline{y}_\alpha(x) = (1-\alpha)(1-x),$$

if $x \leq 1$, we have $\frac{\partial \underline{y}_\alpha(x)}{\partial \alpha} \geq 0, \frac{\partial \overline{y}_\alpha(x)}{\partial \alpha} \leq 0$, and

$\underline{y}_\alpha(x) \leq \overline{y}_\alpha(x)$. Also, since

$\underline{y}'_\alpha(x) - \overline{y}'_\alpha(x) = 2(1-\alpha) \geq 0,$

$\underline{y}''_\alpha(x)$-$\overline{y}''_\alpha(x) = 0.$

(2,1) solution is a valid fuzzy function for $x \leq 1$. Similarly, (2,2) solution is a valid fuzzy function for $x \leq 1$. for (2,1) and (2,2) solutions, since

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$$\underline{y}_1(x) = 2 - 2e^{-\frac{x}{2}} = \overline{y}_1(x) \quad ,$$

$$\underline{y}_1(x) - \underline{y}_\alpha(x) = (1-x)(1-\alpha) = \overline{y}_\alpha(x) - \overline{y}_1(x),$$

(2,1) and (2,2) solutions are symmetric triangular fuzzy numbers for any $x > 0$ time. These results are the same as obtained by Hulya et al 2020[26].

The solution can be represented graphically according to the following figures

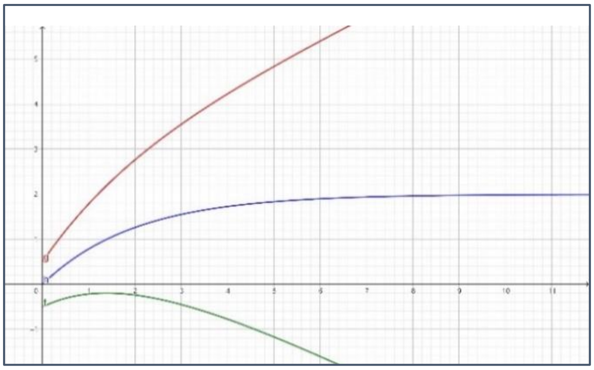

Figure 1. Graphic of (1,1) and (1,2) solutions for $\alpha = 0.5$

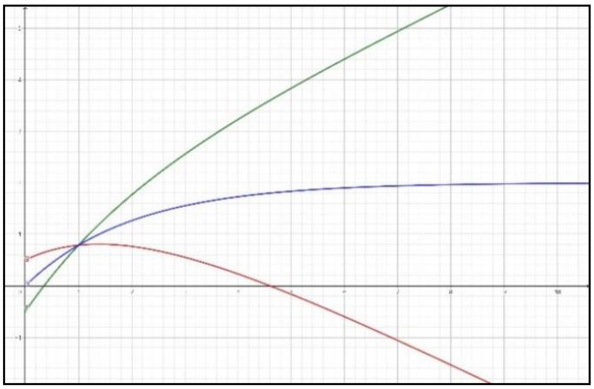

Figure 1. Graphic of (2,1) and (2,2) solutions for $\alpha = 0.5$

Conclusions:

In this paper, we introduced a novel approach to solving second-order fuzzy initial value problems using the fuzzy Sumudu transform and generalized differentiability. Our method offers a direct and efficient solution for fuzzy differential equations with fuzzy initial conditions, overcoming limitations of traditional approaches. By expressing solutions as parametric representations of fuzzy numbers and utilizing the inverse fuzzy Sumudu transform, we provide a comprehensive framework for analyzing the behavior and characteristics of fuzzy initial value problems. The combination of the fuzzy Sumudu transform and generalized differentiability shows great promise in solving fuzzy initial value problems and has practical applications in various domains. Also, we find that the solution obtained by the fuzzy Sumudu transform not only proves to be more concise, but also shows greater flexibility than the solution by the fuzzy Laplace transform. This result emphasizes the practical advantages of using the fuzzy Sumudu transform in solving fuzzy initial-valued differential equations.

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